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The family of quaternionic quasi-unitary Lie algebras and their central extensions

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Abstract. The family of quaternionic quasi-unitary (or quaternionic unitary Cayley–Klein algebras) is described in a unified setting. This family includes the simple algebras $sp(N+1)$ and $sp(p,q)$ in the Cartan series C_{N+1}, as well as many non-semisimple real Lie algebras which can be obtained from these simple algebras by particular contractions. The algebras in this family are realized here in relation with the groups of isometries of quaternionic Hermitian spaces of constant holomorphic curvature. This common framework allows one to perform the study of many properties for all these Lie algebras simultaneously. In this paper the central extensions for all quasi-simple Lie algebras of the quaternionic unitary Cayley–Klein family are shown to be trivial no matter their dimension.

1. Introduction

This paper is devoted to a double purpose. First, it introduces and describes the structure of a family of Lie algebras, the quaternionic quasi-unitary algebras, or quaternionic unitary Cayley–Klein (CK) algebras, which include as simple members the algebras in the Cartan series C_{N+1} which in standard notation are written as $sp(p,q)$, $p+q=N+1$, as well as many non-simple members which can be obtained from the former by a sequence of contractions. The description is also conducted in relation to the symmetric homogeneous spaces (the quaternionic Hermitian spaces of rank one) where these groups act in a natural way.

The second and main purpose is to investigate the Lie algebra cohomology of the algebras in this CK family, in any dimension. These extensions have both a mathematical interest and physical relevance. Therefore, this part of the paper can be considered as a further step in a systematic study of properties of the these families of Lie algebras [1–8], by using a formalism which allows a clear view of the behaviour of these properties under contraction; in physical terms contractions are related to some kind of approximation.

In particular, the central extensions of algebras in the two other main CK families of Lie algebras (the quasi-orthogonal algebras and the two families of quasi-unitary algebras) have been studied in two previous papers, in the general situation and for any dimension [7, 8]. We refer to these works for references and for physical motivations. The knowledge of the second cohomology group for a Lie algebra relies on the general solution of a set of linear equations, but in special cases the calculations may be bypassed by using some general results: for instance, the second cohomology group is trivial for semisimple Lie algebras. But once a contraction is made, the semisimple character disappears, and the contracted algebra *might* have non-trivial central extensions. Instead of finding the general solution for the extension equations on a case-by-case basis, our approach (as developed previously for the quasi-orthogonal algebras [7] and for the quasi-unitary algebras [8]) is to conduct these calculations for a whole family including a large number of algebras simultaneously. In this paper we discuss the 'next' family: the quaternionic quasi-unitary one. The advantages in this approach can be summed up by: (a) it allows us to record, in an easily retrievable form, a large number of results which could be required in applications, both in mathematics and in physics, and (b) it avoids, once and for all, the case-by-case-type computation of the central extensions of algebras included in each family and affords a global view on the interrelations between cohomology and contractions.

Section 2 is devoted to the description of the family of quaternionic unitary CK algebras. We show how to obtain these as graded contractions of the compact algebra $u(N+1,\mathbb{H}) \equiv sp(N+1)$, and we provide some details on their structure. These algebras are associated with the quaternionic Hermitian spaces (of rank one) with metrics of different signatures and with their contractions, so we devote a part of this section to dwell upon these questions. In section 3 the general solution to the central extension problem for these algebras is given. The result obtained is quite simple to state: all the extensions of any algebra in the quaternionic unitary CK family are trivial. This triviality is already known (Whitehead's lemma) for the simple algebras $u(p,q,\mathbb{H}) \equiv sp(p,q)$ in this family, but comes as a surprise for the rather large number of non-semisimple Lie algebras in this CK family, which can be obtained by contracting $u(p,q,\mathbb{H})$. This is also in marked contrast with the results for the central extensions of both the orthogonal and the unitary CK families, where some algebras (particularly the most contracted one) always allow some non-trivial extensions. Finally, some remarks close the paper.

2. The family of quaternionic unitary CK algebras

To begin with we consider the compact real form of the Lie algebra in the Cartan series C_{N+1}. This compact real form can be realized as the Lie algebra of the complex unitary-symplectic group sometimes denoted as $USp(2(N+1))$ [9] but more usually referred to for brevity as the 'symplectic' group, $Sp(N+1)$. The usual convention is to denote this group without any reference to a field to avoid confusion with the true *symplectic* groups over either the reals $Sp(2(N+1),\mathbb{R})$ or over the complex numbers $Sp(2(N+1),\mathbb{C})$; in these last cases the term *symplectic* is properly associated to the symmetry group of an antisymmetric metric. This double use of the name 'symplectic' and of the symbols Sp and sp is rather unfortunate, and following Sudbery [10], we shall change the symbol for one of the families, and use Sq, sq for the unitary-symplectic groups and algebras usually denoted, without any field reference, by Sp, sp.

The group $Sq(N+1) \equiv USp(2(N+1))$ is the intersection of the complex *unitary* group $U(2(N+1),\mathbb{C})$ and the complex *symplectic* group $Sp(2(N+1),\mathbb{C})$:

$$Sq(N+1) \equiv USp(2(N+1)) = U(2(N+1),\mathbb{C}) \cap Sp(2(N+1),\mathbb{C})$$

which is a consequence of the nature of $Sq(N+1)$ as the quaternionic *unitary* group, whose matrices leave invariant a quaternionic definite positive Hermitian metric.

We recall that all other non-compact real forms in the Cartan series C_{N+1} are the real *symplectic* algebra $sp(2(N+1),\mathbb{R})$, and the algebras $sq(p,q)$, $p+q = N+1$, of the quaternionic pseudo-unitary groups $Sq(p,q)$, which allow a realization as

$$Sq(p,q) \equiv USp(2p,2q) = U(2p,2q,\mathbb{C}) \cap Sp(2(N+1),\mathbb{C})$$

and are the groups of pseudo-unitary quaternionic matrices leaving invariant a quaternionic Hermitian metric of signature (p,q).

The Lie algebra $sq(N+1)$ has dimension $2(N+1)^2 + (N+1)$ and is usually realized by $2(N+1) \times 2(N+1)$ complex matrices [9, 11]. The alternative realization of the group $Sq(N+1)$ as a quaternionic unitary matrix group, $Sq(N+1) \equiv U(N+1, \mathbb{H})$ [12], leads to another realization of the Lie algebra $sq(N+1)$ by means of *anti-Hermitian* matrices over the quaternionic skew field $\mathbb{H}$:

$$J_{ab} = -e_{ab} + e_{ba} \qquad M^\alpha_{ab} = i_\alpha(e_{ab} + e_{ba}) \qquad E^\alpha_a = i_\alpha e_{aa} \tag{2.1}$$

where $a < b, a, b = 0, 1, \ldots, N, \alpha = 1, 2, 3; i_1 = i, i_2 = j, i_3 = k$ are the usual quaternionic units, and e_{ab} is the $(N+1) \times (N+1)$ matrix with a single 1 entry in row a, column b. Notice that the matrices J_{ab} and M^α_{ab} are traceless, but the trace of E^α_a is a non-zero pure imaginary quaternion, so the realization is by anti-Hermitian quaternionic matrices whose trace has a zero real part. When quaternions are realized as 2×2 complex matrices (see e.g. [13]) then (2.1) reduces to the usual realization of $sq(N+1)$ by complex matrices $2(N+1) \times 2(N+1)$ which are at the same time complex unitary and complex symplectic; we remark that all these matrices are traceless. In spite of the name quaternionic unitary algebra, the matrices in the vector fundamental representation (2.1) of $sq(N+1)$ are $\mathbb{H}$-*anti-Hermitian*.

The multiplication of quaternionic units is encoded in $i_\alpha i_\beta = -\delta_{\alpha\beta} + \sum_{\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} i_\gamma$ where $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric unit tensor with $\varepsilon_{123} = 1$. This relation allows us to derive the expression for the Lie bracket of two pure quaternionic matrices $X^\alpha = i_\alpha X$, $Y^\beta = i_\beta Y$, where X, Y are real matrices, as

$$[X^\alpha, Y^\beta] = -\delta_{\alpha\beta}[X, Y] + \sum_{\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} i_\gamma \{X, Y\} \tag{2.2}$$

where both the commutator and the anticommutator $\{X, Y\} = XY + YX$ of the real matrices X, Y appear. Using this formula, the commutation relations of $sq(N+1)$ in the basis (2.1) read

$$\begin{aligned}
&[J_{ab}, J_{ac}] = J_{bc} \qquad [J_{ab}, J_{bc}] = -J_{ac} \qquad [J_{ac}, J_{bc}] = J_{ab}\\
&[M^\alpha_{ab}, M^\alpha_{ac}] = J_{bc} \qquad [M^\alpha_{ab}, M^\alpha_{bc}] = J_{ac} \qquad [M^\alpha_{ac}, M^\alpha_{bc}] = J_{ab}\\
&[J_{ab}, M^\alpha_{ac}] = M^\alpha_{bc} \qquad [J_{ab}, M^\alpha_{bc}] = -M^\alpha_{ac} \qquad [J_{ac}, M^\alpha_{bc}] = -M^\alpha_{ab}\\
&[M^\alpha_{ab}, J_{ac}] = -M^\alpha_{bc} \qquad [M^\alpha_{ab}, J_{bc}] = -M^\alpha_{ac} \qquad [M^\alpha_{ac}, J_{bc}] = M^\alpha_{ab}\\
&[J_{ab}, J_{de}] = 0 \qquad [M^\alpha_{ab}, M^\alpha_{de}] = 0 \qquad [J_{ab}, M^\alpha_{de}] = 0\\
&[J_{ab}, E^\alpha_d] = (\delta_{ad} - \delta_{bd})M^\alpha_{ab} \qquad [M^\alpha_{ab}, E^\alpha_d] = -(\delta_{ad} - \delta_{bd})J_{ab}\\
&[J_{ab}, M^\alpha_{ab}] = 2(E^\alpha_b - E^\alpha_a) \qquad [E^\alpha_a, E^\alpha_b] = 0
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&[M^\alpha_{ab}, M^\beta_{ac}] = \varepsilon_{\alpha\beta\gamma} M^\gamma_{bc} \qquad [M^\alpha_{ab}, M^\beta_{bc}] = \varepsilon_{\alpha\beta\gamma} M^\gamma_{ac} \qquad [M^\alpha_{ac}, M^\beta_{bc}] = \varepsilon_{\alpha\beta\gamma} M^\gamma_{ab}\\
&[M^\alpha_{ab}, M^\beta_{de}] = 0 \qquad [M^\alpha_{ab}, M^\beta_{ab}] = 2\varepsilon_{\alpha\beta\gamma}(E^\gamma_a + E^\gamma_b)\\
&[M^\alpha_{ab}, E^\beta_d] = (\delta_{ad} + \delta_{bd})\varepsilon_{\alpha\beta\gamma} M^\gamma_{ab} \qquad [E^\alpha_a, E^\beta_b] = 2\delta_{ab}\varepsilon_{\alpha\beta\gamma} E^\gamma_a
\end{aligned} \tag{2.4}$$

where hereafter the following notational conventions are assumed:

- Whenever three indices a, b, c appear, they are always assumed to verify $a < b < c$.
- Whenever three indices a, b, d appear, $a < b$ is assumed but the index d is arbitrary, and it might coincide with either a or b.
- Whenever four indices a, b, d, e appear, $a < b$, $d < e$ and all of them are assumed to be different.
- Whenever three quaternionic indices α, β, γ appear, they are also assumed to be different (so they are always some permutation of 1, 2, 3).
- There is not any implied sum over repeated indices; in particular there is no sum in γ in expressions like $\varepsilon_{\alpha\beta\gamma} X^\gamma$.

This matrix realization of the Lie algebra $sq(N+1)$ displays clearly the existence of several subalgebras. On the one hand, the $\frac{1}{2}N(N+1)$ generators J_{ab} $(a, b = 0, 1, \ldots, N)$ close an orthogonal algebra $so(N+1)$ whose non-zero commutation rules are written in the first row of (2.3). On the other hand, for each *fixed* $\alpha = 1, 2, 3$, the $(N+1)^2$ generators $\{J_{ab}, M^\alpha_{ab}, E^\alpha_a\}$ $(a, b = 0, 1, \ldots, N;\ a < b)$ give rise to an algebra isomorphic to the unitary algebra $u(N+1)$ with commutators given by (2.3); these subalgebras we denote as $u^\alpha(N+1)$. Hence $sq(N+1)$ contains *three* subalgebras isomorphic to $u(N+1)$, whose intersection is a subalgebra $so(N+1)$.

The family of algebras we study in this paper can be obtained as graded contractions [14, 15] from $sq(N+1)$. The algebra $sq(N+1)$ can be endowed with a grading by a group $\mathbb{Z}_2^{\otimes N}$ constituted by 2^N involutive automorphisms $S_{\mathcal{S}}$ defined by

$$
\begin{aligned}
&S_{\mathcal{S}} J_{ab} = (-1)^{\chi_{\mathcal{S}}(a)+\chi_{\mathcal{S}}(b)} J_{ab} \\
&S_{\mathcal{S}} M^\alpha_{ab} = (-1)^{\chi_{\mathcal{S}}(a)+\chi_{\mathcal{S}}(b)} M^\alpha_{ab} \qquad S_{\mathcal{S}} E^\alpha_a = E^\alpha_a \qquad \alpha = 1, 2, 3
\end{aligned}
\tag{2.5}
$$

where $\mathcal{S}$ denotes any subset of the set of indices $\{0, 1, \ldots, N\}$, and $\chi_{\mathcal{S}}(a)$ denotes the characteristic function over $\mathcal{S}$. A particular solution of the $\mathbb{Z}_2^{\otimes N}$ graded contractions of $sq(N+1)$ leads to a family of Lie algebras which are called quaternionic unitary CK algebras or quaternionic quasi-unitary Lie algebras [2, 3]. This family comprises the simple quaternionic unitary and pseudo-unitary algebras $sq(p, q)$ $(p+q = N+1)$ in the Cartan series C_{N+1} as well as many non-simple real Lie algebras which can be obtained from the former by contractions. Collectively, all these algebras preserve some properties related to simplicity, so they belong to the class of so-called 'quasi-simple' Lie algebras [16, 17], which explains the use of the prefix quasi in their name. Overall, this is very similar to the situation of the families of quasi-orthogonal algebras (with $so(N+1)$ as the initial Lie algebra [1, 4]) or to the families of quasi-unitary or quasi-special unitary algebras over the complex numbers (starting from either $u(N+1)$ or $su(N+1)$ [8]).

The quaternionic unitary CK algebras can be described by means of N real coefficients ω_a $(a = 1, \ldots, N)$ and are denoted collectively as $sq_{\omega_1,\ldots,\omega_N}(N+1)$, or in an abbreviated form, as $sq_\omega(N+1)$ where ω stands for $\omega = (\omega_1, \ldots, \omega_N)$. We introduce the two-index coefficients ω_{ab} defined by

$$
\omega_{ab} := \omega_{a+1}\omega_{a+2}\ldots\omega_b \qquad a, b = 0, 1, \ldots, N \qquad a < b \qquad \omega_{aa} := 1
\tag{2.6}
$$

and the commutation relations of the generic CK algebra in the family $sq_\omega(N+1)$ turn out to be [2]

$$
\begin{aligned}
&[J_{ab}, J_{ac}] = \omega_{ab} J_{bc} \qquad [J_{ab}, J_{bc}] = -J_{ac} \qquad [J_{ac}, J_{bc}] = \omega_{bc} J_{ab} \\
&[M^\alpha_{ab}, M^\alpha_{ac}] = \omega_{ab} J_{bc} \qquad [M^\alpha_{ab}, M^\alpha_{bc}] = J_{ac} \qquad [M^\alpha_{ac}, M^\alpha_{bc}] = \omega_{bc} J_{ab} \\
&[J_{ab}, M^\alpha_{ac}] = \omega_{ab} M^\alpha_{bc} \qquad [J_{ab}, M^\alpha_{bc}] = -M^\alpha_{ac} \qquad [J_{ac}, M^\alpha_{bc}] = -\omega_{bc} M^\alpha_{ab} \\
&[M^\alpha_{ab}, J_{ac}] = -\omega_{ab} M^\alpha_{bc} \qquad [M^\alpha_{ab}, J_{bc}] = -M^\alpha_{ac} \qquad [M^\alpha_{ac}, J_{bc}] = \omega_{bc} M^\alpha_{ab} \\
&[J_{ab}, J_{de}] = 0 \qquad [M^\alpha_{ab}, M^\alpha_{de}] = 0 \qquad [J_{ab}, M^\alpha_{de}] = 0 \\
&[J_{ab}, E^\alpha_d] = (\delta_{ad} - \delta_{bd}) M^\alpha_{ab} \qquad [M^\alpha_{ab}, E^\alpha_d] = -(\delta_{ad} - \delta_{bd}) J_{ab} \\
&[J_{ab}, M^\alpha_{ab}] = 2\omega_{ab}(E^\alpha_b - E^\alpha_a) \qquad [E^\alpha_a, E^\alpha_b] = 0
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&[M^\alpha_{ab}, M^\beta_{ac}] = \omega_{ab}\varepsilon_{\alpha\beta\gamma} M^\gamma_{bc} \qquad [M^\alpha_{ab}, M^\beta_{bc}] = \varepsilon_{\alpha\beta\gamma} M^\gamma_{ac} \qquad [M^\alpha_{ac}, M^\beta_{bc}] = \omega_{bc}\varepsilon_{\alpha\beta\gamma} M^\gamma_{ab} \\
&[M^\alpha_{ab}, M^\beta_{de}] = 0 \qquad [M^\alpha_{ab}, M^\beta_{ab}] = 2\omega_{ab}\varepsilon_{\alpha\beta\gamma}(E^\gamma_a + E^\gamma_b) \\
&[M^\alpha_{ab}, E^\beta_d] = (\delta_{ad} + \delta_{bd})\varepsilon_{\alpha\beta\gamma} M^\gamma_{ab} \qquad [E^\alpha_a, E^\beta_b] = 2\delta_{ab}\varepsilon_{\alpha\beta\gamma} E^\gamma_a
\end{aligned}
\tag{2.8}
$$

where we adhere to the notational conventions given after (2.4).

The pattern of subalgebras previously discussed for the compact form $sq(N+1)$ clearly holds for any member of the complete family. The quaternionic unitary CK algebra $sq_\omega(N+1)$

also contains as Lie subalgebras an orthogonal CK algebra $so_\omega(N+1)$ [2, 7] and *three* unitary CK algebras $u^\alpha_\omega(N+1)$ [2, 8] where $\alpha = 1, 2, 3$; the commutation relations of the former correspond to the first row of (2.7) and those of the latter are given by (2.7) (for an index α fixed). Hence we find the sequence

$$so_\omega(N+1) \subset u^\alpha_\omega(N+1) \subset sq_\omega(N+1). \tag{2.9}$$

2.1. The quaternionic unitary CK groups

The matrix realization (2.1) allows a natural interpretation of the quaternionic unitary CK algebras as the Lie algebras of the motion groups of the homogeneous symmetric spaces with a quaternionic Hermitian metric (the two-point homogeneous spaces of quaternionic type and rank one). Let us consider the space $\mathbb{H}^{N+1}$ endowed with a Hermitian (sesqui)linear form $\langle\cdot|\cdot\rangle_\omega : \mathbb{H}^{N+1} \times \mathbb{H}^{N+1} \to \mathbb{H}$ defined by

$$\langle \boldsymbol{a}|\boldsymbol{b}\rangle_\omega := \bar{a}^0 b^0 + \bar{a}^1\omega_1 b^1 + \bar{a}^2\omega_1\omega_2 b^2 + \cdots + \bar{a}^N\omega_1\ldots\omega_N b^N = \sum_{i=0}^{N} \bar{a}^i \omega_{0i} b^i \tag{2.10}$$

where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{H}^{N+1}$ and $\bar{a}^i$ means the quaternionic conjugation of the component a^i. For the moment, we assume that we are in the generic case with all $\omega_a \neq 0$. The underlying metric is provided by the matrix

$$\mathcal{I}_\omega = \mathrm{diag}(1, \omega_{01}, \omega_{02}, \ldots, \omega_{0N}) = \mathrm{diag}(1, \omega_1, \omega_1\omega_2, \ldots, \omega_1\ldots\omega_N) \tag{2.11}$$

and the CK group $Sq_{\omega_1,\ldots,\omega_N}(N+1) \equiv Sq_\omega(N+1)$ is defined as the group of linear isometries of this Hermitian metric over a quaternionic space. Thus the isometry condition for an element U of the Lie group

$$\langle U\boldsymbol{a}|U\boldsymbol{b}\rangle_\omega = \langle \boldsymbol{a}|\boldsymbol{b}\rangle_\omega \qquad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{H}^{N+1} \tag{2.12}$$

leads to the following relation:

$$U^\dagger \mathcal{I}_\omega U = \mathcal{I}_\omega \qquad \forall U \in Sq_\omega(N+1) \tag{2.13}$$

which for the Lie algebra implies

$$X^\dagger \mathcal{I}_\omega + \mathcal{I}_\omega X = 0 \qquad \forall X \in sq_\omega(N+1). \tag{2.14}$$

From this equation, it is clear that the quaternionic unitary CK algebra is generated by the following $(N+1)\times(N+1)$ $\mathcal{I}_\omega$-anti-Hermitian matrices over $\mathbb{H}$ (cf (2.1)):

$$J_{ab} = -\omega_{ab}e_{ab} + e_{ba} \qquad M^\alpha_{ab} = i_\alpha(\omega_{ab}e_{ab} + e_{ba}) \qquad E^\alpha_a = i_\alpha e_{aa}. \tag{2.15}$$

These matrices can be checked to satisfy the commutation relations (2.7) and (2.8).

When any of the constants ω_a are equal to zero, the set of linear isometries of the Hermitian metric over the quaternions (2.12) is larger than the group generated by (2.15), though in these cases there exist additional geometric structures in $\mathbb{H}^{N+1}$, which are related to the existence of invariant foliations, and the proper definition of the automorphism group for these structures leads again to the matrix Lie algebra generated by (2.15) with commutation relations (2.7) and (2.8).

The action of the group $Sq_\omega(N+1)$ in $\mathbb{H}^{N+1}$ is not transitive, and the 'sphere' with equation

$$\langle \boldsymbol{x}|\boldsymbol{x}\rangle_\omega := \sum_{i=0}^{N} \bar{x}^i \omega_{0i} x^i = 1 \tag{2.16}$$

is stable. However, if we take $O = (1, 0, \ldots, 0)$ as a reference point in this sphere, the realization (2.15) shows that the isotropy subgroup of O is $Sq_{\omega_2,\omega_3,\ldots,\omega_N}(N)$, and the isotropy subgroup of the *ray* of O is $Sq(1) \otimes Sq_{\omega_2,\omega_3,\ldots,\omega_N}(N)$ (note that the quaternions being non-commutative, a choice for left or right multiplication for scalars is required). Here the algebra $sq(1)$ of the subgroup $Sq(1)$ can be identified with the Lie algebra of automorphisms of the quaternions, generated by the three matrices

$$I^\alpha = i_\alpha \sum_{a=0}^{N} e_{aa} \qquad \alpha = 1, 2, 3 \tag{2.17}$$

which can be identified with the three quaternionic units. We note in passing that these are the elements of the Lie algebra which are unavoidably realized by matrices with non-zero pure imaginary trace, as all the generators E_a^α can be expressed in terms of zero trace combinations (say $B_l^\alpha \equiv E_{l-1}^\alpha - E_l^\alpha$, $l = 1, \ldots, N$) and the three I^α. In this way we find the quaternionic Hermitian homogeneous spaces as associated with the quaternionic unitary family of CK groups:

$$Sq_{\omega_1,\omega_2,\omega_3,\ldots,\omega_N}(N+1)/(Sq(1) \otimes Sq_{\omega_2,\omega_3,\ldots,\omega_N}(N)). \tag{2.18}$$

For fixed $\omega_1, \omega_2, \omega_3, \ldots, \omega_N$ this space, which has real dimension $4N$, has a natural real quadratic metric (either Riemannian, pseudo-Riemannian or degenerate 'Riemannian'), coming from the real part of the quaternionic Hermitian product in the ambient space. At the origin and in an adequate basis, this metric is given by the diagonal matrix with entries $(1, \omega_2, \omega_2\omega_3, \ldots, \omega_2 \ldots \omega_N)$, each entry repeated four times. The three well known Hermitian elliptic, euclidean and hyperbolic quaternionic spaces, of constant holomorphic curvature $4K$ (either $K > 0$, $K = 0$ and $K < 0$, respectively) appear in this family as associated to the special values $\omega_1 = K$ and $\omega_2 = \omega_3 = \cdots = \omega_N = 1$, where the metric is Riemannian (definite positive). All CK Hermitian spaces of quaternionic type with $\omega_1 = K$ have constant holomorphic curvature $4K$ and the signature (and/or the eventual degeneracy) of the metric is determined by the remaining constants $\omega_2, \omega_3, \ldots, \omega_N$. When all these constants are different from zero, but some are negative, the metric is pseudo-Riemannian (indefinite and not degenerate), and when some of the constants $\omega_2, \omega_3, \ldots, \omega_N$ vanish the metric is degenerate.

2.2. Structure of the quaternionic unitary CK algebras

As each real coefficient ω_a can be positive, negative or zero, the quaternionic unitary CK family $sq_\omega(N+1)$ includes 3^N Lie algebras. Semisimple algebras appear when all the coefficients ω_a are different from zero: these are the algebras $sq(p,q)$ in the Cartan series C_{N+1}, where p and q $(p + q = N + 1)$ are the number of positive and negative terms in the matrix $\mathcal{I}_\omega$ (2.11). If we set all $\omega_a = 1$ we recover the initial compact algebra $sq(N+1)$. When one or more coefficients ω_a vanish the CK algebra turns out to be a non-semisimple Lie algebra; the vanishing of one (or several) coefficient ω_a is equivalent to performing an (or series of) Inönü–Wigner contraction [18, 19].

Some of the quaternionic unitary CK algebras are isomorphic; for instance, the isomorphism

$$sq_{\omega_1,\omega_2,\ldots,\omega_{N-1},\omega_N}(N+1) \simeq sq_{\omega_N,\omega_{N-1},\ldots,\omega_2,\omega_1}(N+1) \tag{2.19}$$

(that interchanges $\omega_{ab} \leftrightarrow \omega_{N-b,N-a}$) is provided by the map

$$\begin{array}{ll} J_{ab} \to J'_{ab} = -J_{N-b,N-a} & \\ M^1_{ab} \to M'^1_{ab} = -M^2_{N-b,N-a} & E^1_a \to E'^1_a = -E^2_{N-a} \\ M^2_{ab} \to M'^2_{ab} = -M^1_{N-b,N-a} & E^2_a \to E'^2_a = -E^1_{N-a} \\ M^3_{ab} \to M'^3_{ab} = -M^3_{N-b,N-a} & E^3_a \to E'^3_a = -E^3_{N-a}. \end{array} \tag{2.20}$$

Each algebra in the family of quaternionic unitary CK algebras has many subalgebras isomorphic to orthogonal, unitary, or special unitary CK algebras, as well as many subalgebras isomorphic to quaternionic unitary algebras in the family $sq_\omega(M+1)$ with $M < N$. A clear way to describe this is to denote by X_{ab} the four generators $\{J_{ab}, M^\alpha_{ab}\}$ $(\alpha = 1, 2, 3)$, by E_a the set of three generators E^α_a, and arrange the basis generators of $sq_\omega(N+1)$ as follows:

$$
\begin{array}{ccccc|cccc}
E_0 & X_{01} & X_{02} & \ldots & X_{0\,a-1} & X_{0a} & X_{0\,a+1} & \ldots & X_{0N} \\
 & E_1 & X_{12} & \ldots & X_{1\,a-1} & X_{1a} & X_{1\,a+1} & \ldots & X_{1N} \\
 & & \ddots & & \vdots & \vdots & \vdots & & \vdots \\
 & & & E_{a-2} & X_{a-2\,a-1} & X_{a-2\,a} & X_{a-2\,a+1} & \ldots & X_{a-2\,N} \\
 & & & & E_{a-1} & X_{a-1\,a} & X_{a-1\,a+1} & \ldots & X_{a-1\,N} \\
\cline{6-9}
 & & & & & E_a & X_{a\,a+1} & \ldots & X_{aN} \\
 & & & & & & \ddots & & \vdots \\
 & & & & & & & E_{N-1} & X_{N-1\,N} \\
 & & & & & & & & E_N .
\end{array}
$$

A Cartan subalgebra is made up of the $N+1$ generators $E^3_0, E^3_1, \ldots, E^3_N$ (in the outermost diagonal). In this arrangement the generators to the left and below the rectangle span subalgebras $sq_{\omega_1,\ldots,\omega_{a-1}}(a)$ and $sq_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$, respectively, while the generators inside the rectangle do not span a subalgebra unless $\omega_a = 0$ (and in this case this is an Abelian subalgebra). The unitary subalgebras $u^\alpha_\omega(N+1)$ appear in a similar way by keeping only J_{ab}, a single M^α_{ab} out of each X_{ab} and a single E^α_a out of each set E_a (for a fixed α). By keeping only J_{ab} we get the $so_\omega(N+1)$ subalgebra.

If a coefficient $\omega_a = 0$, then the contracted algebra has a semidirect structure

$$sq_{\omega_1,\ldots,\omega_{a-1},\omega_a=0,\omega_{a+1},\ldots,\omega_N}(N+1) \equiv t \odot (sq_{\omega_1,\ldots,\omega_{a-1}}(a) \oplus sq_{\omega_{a+1},\ldots,\omega_N}(N+1-a)) \tag{2.21}$$

where t is spanned by the generators inside the rectangle (it is an Abelian subalgebra of dimension $4a(N+1-a)$), while $sq_{\omega_1,\ldots,\omega_{a-1}}(a)$ and $sq_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$ are two quaternionic unitary CK subalgebras spanned by the generators in the triangles to the left and below the rectangle. When there are several coefficients $\omega_a = 0$ the contracted algebra has simultaneously several semidirect structures (2.21).

Notice that when $\omega_1 = 0$ the contracted algebra has the structure

$$sq_{0,\omega_2,\ldots,\omega_N}(N+1) \equiv t_{4N} \odot (sq(1) \oplus sq_{\omega_2,\ldots,\omega_N}(N)) \tag{2.22}$$

and here the subindex $4N$ in t is the real dimension of the flat homogeneous space (2.18) which can be identified with $\mathbb{H}^N$ endowed with a flat metric given, over $\mathbb{H}$, by the diagonal matrix $(1, \omega_2, \omega_2\omega_3, \ldots, \omega_2\omega_3\ldots\omega_N)$; when all these are positive this Lie algebra can be called inhomogeneous quaternionic unitary algebra $isq(N)$.

3. Central extensions

After having described the structure of the quaternionic unitary CK algebras, we now turn to the second goal of this paper: to give a complete description of all possible central extensions of the algebras in the quaternionic unitary CK family. The outcome of this study is simple to state: in any dimension, and for all quaternionic unitary CK algebras—no matter of how many ω_a are equal or different from zero—there are no non-trivial central extensions.

For any r-dimensional Lie algebra with generators $\{X_1, \ldots, X_r\}$ and structure constants C^k_{ij}, a generic central extension by the one-dimensional algebra generated by Ξ will have commutation relations given by

$$[X_i, X_j] = \sum_{k=1}^{r} C_{ij}^k X_k + \xi_{ij}\,\Xi \qquad [\Xi, X_i] = 0. \tag{3.1}$$

The extension coefficients or central charges ξ_{ij} must be antisymmetric in the indices i, j, $\xi_{ji} = -\xi_{ij}$ and must fulfil the following conditions coming from the Jacobi identities for the generators X_i, X_j, X_l in the extended Lie algebra:

$$\sum_{k=1}^{r} (C_{ij}^k \xi_{kl} + C_{jl}^k \xi_{ki} + C_{li}^k \xi_{kj}) = 0. \tag{3.2}$$

If for a set of arbitrary real numbers μ_i we perform a change of generators:

$$X_i \to X_i' = X_i + \mu_i\,\Xi \tag{3.3}$$

the commutation rules for the generators $\{X_i'\}$ are given by the expressions (3.1) with a new set of $\xi_{ij}' = \xi_{ij} - \sum_{k=1}^{r} C_{ij}^k \mu_k$, where $\delta\mu(X_i, X_j) = \sum_{k=1}^{r} C_{ij}^k \mu_k$ is the two-coboundary generated by μ. Extension coefficients differing by a two-coboundary correspond to equivalent extensions; and those extension coefficients which are a two-coboundary $\xi_{ij} = -\sum_{k=1}^{r} C_{ij}^k \mu_k$ correspond to trivial extensions; the classes of equivalence of non-trivial two-cocycles determine the second cohomology group of the Lie algebra.

3.1. Central extensions of the unitary CK subalgebras

In order to simplify further computations, we first state the result about the structure of the central extensions of the unitary CK algebra $u_\omega(N+1)$[8], which will naturally appear when studying the extensions of the quaternionic unitary CK algebras, because each $sq_\omega(N+1)$ contains three such unitary CK subalgebras.

Theorem 3.1. *The commutation relations of any central extension $\overline{u}_\omega^\alpha(N+1)$ of the unitary CK algebra $u_\omega^\alpha(N+1)$ with generators $\{J_{ab}, M_{ab}^\alpha, E_a^\alpha\}$ ($a, b = 0, 1, \ldots, N$ and quaternionic index α fixed) by the one-dimensional algebra generated by Ξ are*

$$\begin{aligned}
&[J_{ab}, J_{ac}] = \omega_{ab}(J_{bc} + h_{bc}^\alpha\,\Xi) \qquad [M_{ab}^\alpha, M_{ac}^\alpha] = \omega_{ab}(J_{bc} + h_{bc}^\alpha\,\Xi)\\
&[J_{ab}, J_{bc}] = -(J_{ac} + h_{ac}^\alpha\,\Xi) \qquad [M_{ab}^\alpha, M_{bc}^\alpha] = J_{ac} + h_{ac}^\alpha\,\Xi\\
&[J_{ac}, J_{bc}] = \omega_{bc}(J_{ab} + h_{ab}^\alpha\,\Xi) \qquad [M_{ac}^\alpha, M_{bc}^\alpha] = \omega_{bc}(J_{ab} + h_{ab}^\alpha\,\Xi)\\
&[J_{ab}, J_{de}] = 0 \qquad [M_{ab}^\alpha, M_{de}^\alpha] = 0\\
&[J_{ab}, M_{ac}^\alpha] = \omega_{ab}(M_{bc}^\alpha + g_{bc}^\alpha\,\Xi) \qquad [M_{ab}^\alpha, J_{ac}] = -\omega_{ab}(M_{bc}^\alpha + g_{bc}^\alpha\,\Xi)\\
&[J_{ab}, M_{bc}^\alpha] = -(M_{ac}^\alpha + g_{ac}^\alpha\,\Xi) \qquad [M_{ab}^\alpha, J_{bc}] = -(M_{ac}^\alpha + g_{ac}^\alpha\,\Xi)\\
&[J_{ac}, M_{bc}^\alpha] = -\omega_{bc}(M_{ab}^\alpha + g_{ab}^\alpha\,\Xi) \qquad [M_{ac}^\alpha, J_{bc}] = \omega_{bc}(M_{ab}^\alpha + g_{ab}^\alpha\,\Xi)\\
&[J_{ab}, E_d^\alpha] = (\delta_{ad} - \delta_{bd})(M_{ab}^\alpha + g_{ab}^\alpha\,\Xi) \qquad [J_{ab}, M_{de}^\alpha] = 0\\
&[M_{ab}^\alpha, E_d^\alpha] = -(\delta_{ad} - \delta_{bd})(J_{ab} + h_{ab}^\alpha\,\Xi)
\end{aligned} \tag{3.4}$$

$$[J_{ab}, M_{ab}^\alpha] = 2\omega_{ab}(E_b^\alpha - E_a^\alpha) + f_{ab}^\alpha\,\Xi \qquad [E_a^\alpha, E_b^\alpha] = e_{a,b}^\alpha\,\Xi \qquad a < b \tag{3.5}$$

where

$$f_{ab}^\alpha = \sum_{s=a+1}^{b} \omega_{a\,s-1}\omega_{sb} f_{s-1\,s}^\alpha. \tag{3.6}$$

The extension is characterized by the following types of extension coefficients:

Type I. $N(N+1)/2$ coefficients g_{ab}^α and $N(N+1)/2$ coefficients h_{ab}^α ($a < b$ and $a, b = 0, 1, \ldots, N$).

Type II. N coefficients $f^{\alpha}_{a-1\,a}$ $(a = 1, \ldots, N)$.

Type III. $N(N+1)/2$ coefficients $e^{\alpha}_{a,b}$ $(a < b$ and $a, b = 0, 1, \ldots, N)$, satisfying

$$\omega_{ab} e^{\alpha}_{a,b} = 0 \qquad \omega_{ab}(e^{\alpha}_{a,c} - e^{\alpha}_{b,c}) = 0 \qquad \omega_{bc}(e^{\alpha}_{a,b} - e^{\alpha}_{a,c}) = 0 \qquad a < b < c. \tag{3.7}$$

This theorem expresses the results previously obtained in [8] but in a different basis (we are using here a different set of Cartan generators) so that we use another notation for the extension coefficients.

The extension coefficients are classed into types according as their behaviour under contraction. All type I coefficients correspond to central extensions which are trivial for all the unitary CK algebras, no matter of how many coefficients ω_a are equal to zero, since they can be removed at once by means of the redefinitions

$$J_{ab} \to J_{ab} + h^{\alpha}_{ab}\,\Xi \qquad M^{\alpha}_{ab} \to M^{\alpha}_{ab} + g^{\alpha}_{ab}\,\Xi. \tag{3.8}$$

Each type II coefficient $f^{\alpha}_{a-1\,a}$ gives rise to a non-trivial extension if $\omega_a = 0$ and to a trivial one otherwise. That is, these extensions become non-trivial through the contraction and they behave as pseudoextensions [20, 21]. On the other hand, when a type III coefficient $e^{\alpha}_{a,b}$ is non-zero, the extension that it determines is always non-trivial so that it cannot appear through a pseudoextension process. Therefore, the only extensions which can be non-trivial for a given algebra in the CK family $\overline{u}_{\omega}(N+1)$ are those appearing in the Lie brackets (3.5).

We also recall that the dimension of the second cohomology group of a unitary CK algebra $u_{\omega}(N+1)$ with n coefficients ω_a equal to zero is

$$\dim(H^2(u_{\omega}(N+1), \mathbb{R})) = n + \frac{n(n+1)}{2} = \frac{n(n+3)}{2} \tag{3.9}$$

where the first term n corresponds to the extension coefficients $f^{\alpha}_{a-1\,a}$ and the second term $\frac{n(n+1)}{2}$ to the extensions determined by $e^{\alpha}_{a,b}$.

3.2. Central extensions of the quaternionic unitary CK algebras

In what follows we determine the non-trivial extension coefficients ξ_{ij} for a generic centrally extended quaternionic unitary CK algebra $\overline{sq}_{\omega}(N+1)$ (3.1) by solving the Jacobi identities (3.2).

First, we consider a generic extended unitary CK subalgebra, say $\overline{u}^{1}_{\omega}(N+1)$, spanned by the generators $\{J_{ab}, M^1_{ab}, E^1_a, \Xi\}$ $(a, b = 0, 1, \ldots, N;\ a < b)$ with pure quaternionic index equal to 1. It is clear that the set of Jacobi identities involving only these generators lead to the results given in the theorem 3.1. Hence, we find the commutation relations (3.4) and (3.5) with extension coefficients denoted g^1_{ab}, h^1_{ab}, f^1_{ab} and $e^1_{a,b}$; we apply the redefinitions (cf (3.8))

$$J_{ab} \to J_{ab} + h^1_{ab}\,\Xi \qquad M^1_{ab} \to M^1_{ab} + g^1_{ab}\,\Xi \tag{3.10}$$

and the Lie brackets of $\overline{u}^{1}_{\omega}(N+1) \subset \overline{sq}_{\omega}(N+1)$ turn out to be

$$\begin{aligned}
&[J_{ab}, J_{ac}] = \omega_{ab} J_{bc} \qquad [J_{ab}, J_{bc}] = -J_{ac} \qquad [J_{ac}, J_{bc}] = \omega_{bc} J_{ab}\\
&[M^1_{ab}, M^1_{ac}] = \omega_{ab} J_{bc} \qquad [M^1_{ab}, M^1_{bc}] = J_{ac} \qquad [M^1_{ac}, M^1_{bc}] = \omega_{bc} J_{ab}\\
&[J_{ab}, M^1_{ac}] = \omega_{ab} M^1_{bc} \qquad [J_{ab}, M^1_{bc}] = -M^1_{ac} \qquad [J_{ac}, M^1_{bc}] = -\omega_{bc} M^1_{ab}\\
&[M^1_{ab}, J_{ac}] = -\omega_{ab} M^1_{bc} \qquad [M^1_{ab}, J_{bc}] = -M^1_{ac} \qquad [M^1_{ac}, J_{bc}] = \omega_{bc} M^1_{ab}\\
&[J_{ab}, J_{de}] = 0 \qquad [M^1_{ab}, M^1_{de}] = 0 \qquad [J_{ab}, M^1_{de}] = 0\\
&[J_{ab}, E^1_d] = (\delta_{ad} - \delta_{bd}) M^1_{ab} \qquad [M^1_{ab}, E^1_d] = -(\delta_{ad} - \delta_{bd}) J_{ab}
\end{aligned} \tag{3.11}$$

$$[J_{ab}, M^1_{ab}] = 2\omega_{ab}(E^1_b - E^1_a) + f^1_{ab}\,\Xi \qquad [E^1_a, E^1_b] = e^1_{a,b}\,\Xi \qquad a < b. \tag{3.12}$$

The two remaining extended unitary CK subalgebras $\overline{u}^{\lambda}_{\omega}(N+1) \subset \overline{sq}_{\omega}(N+1)$ with $\lambda = 2, 3$ are generated by $\{J_{ab}, M^{\lambda}_{ab}, E^{\lambda}_a, \Xi\}$ (hereafter we shall reserve λ to stand exclusively for the quaternionic indices $\lambda = 2, 3$, whereas α, β, γ are allowed to take on any value 1, 2, 3). The subalgebras $\overline{u}^{\lambda}_{\omega}(N+1)$ have generic extended Lie brackets (as (3.1)) except for the common orthogonal CK subalgebra $so_{\omega}(N+1)$ spanned by the generators $\{J_{ab}\}$ which is non-extended and whose Lie brackets are already written in (3.11). For the two remaining unitary subalgebras, we have already used up the redefinition concerning the common generators in $so_{\omega}(N+1)$, so we cannot directly apply the results of the theorem 3.1 and we have to compute their corresponding Jacobi identities by taking into account that initially both contain a non-extended $so_{\omega}(N+1)$. As the equations so obtained are similar to those written in detail in [8] we omit them and give the final result. The extension coefficients that appear are denoted g^{λ}_{ab}, $h^{\lambda}_{a\,a+1}$, f^{λ}_{ab} and $e^{\lambda}_{a,b}$, for $\lambda = 2, 3$; the Lie brackets of $\overline{u}^{\lambda}_{\omega}(N+1)$ read

$$
\begin{aligned}
&[M^{\lambda}_{ab}, M^{\lambda}_{ac}] = \omega_{ab} J_{bc} \qquad [M^{\lambda}_{ab}, M^{\lambda}_{bc}] = J_{ac} \qquad [M^{\lambda}_{ac}, M^{\lambda}_{bc}] = \omega_{bc} J_{ab} \\
&[J_{ab}, M^{\lambda}_{ac}] = \omega_{ab}(M^{\lambda}_{bc} + g^{\lambda}_{bc}\Xi) \qquad [M^{\lambda}_{ab}, J_{ac}] = -\omega_{ab}(M^{\lambda}_{bc} + g^{\lambda}_{bc}\Xi) \\
&[J_{ab}, M^{\lambda}_{bc}] = -(M^{\lambda}_{ac} + g^{\lambda}_{ac}\Xi) \qquad [M^{\lambda}_{ab}, J_{bc}] = -(M^{\lambda}_{ac} + g^{\lambda}_{ac}\Xi) \\
&[J_{ac}, M^{\lambda}_{bc}] = -\omega_{bc}(M^{\lambda}_{ab} + g^{\lambda}_{ab}\Xi) \qquad [M^{\lambda}_{ac}, J_{bc}] = \omega_{bc}(M^{\lambda}_{ab} + g^{\lambda}_{ab}\Xi) \\
&[J_{ab}, M^{\lambda}_{de}] = 0 \qquad [M^{\lambda}_{ab}, M^{\lambda}_{de}] = 0 \\
&[J_{ab}, E^{\lambda}_{d}] = (\delta_{ad} - \delta_{bd})(M^{\lambda}_{ab} + g^{\lambda}_{ab}\Xi) \\
&[M^{\lambda}_{ab}, E^{\lambda}_{d}] = -(\delta_{ad} - \delta_{bd}) J_{ab} \qquad b > a+1 \\
&[M^{\lambda}_{a\,a+1}, E^{\lambda}_{d}] = -(\delta_{ad} - \delta_{a+1\,d})(J_{a\,a+1} + h^{\lambda}_{a\,a+1}\Xi)
\end{aligned}
\tag{3.13}
$$

$$
[J_{ab}, M^{\lambda}_{ab}] = 2\omega_{ab}(E^{\lambda}_{b} - E^{\lambda}_{a}) + f^{\lambda}_{ab}\Xi \qquad [E^{\lambda}_{a}, E^{\lambda}_{b}] = e^{\lambda}_{a,b}\Xi \qquad a < b. \tag{3.14}
$$

The coefficients f^{λ}_{ab} and $e^{\lambda}_{a,b}$ ($\lambda = 2, 3$) are characterized by theorem 3.1 (see (3.6) and (3.7)), while the extensions $h^{\lambda}_{a\,a+1}$ are subjected to the relations

$$
\omega_a h^{\lambda}_{a\,a+1} = 0 \qquad \omega_{a+2} h^{\lambda}_{a\,a+1} = 0. \tag{3.15}
$$

Notice that now the coefficients h^{λ}_{ab} with $b > a+1$ are zero (this is a direct consequence of the presence of the non-extended $so_{\omega}(N+1)$). We now apply the redefinitions given by

$$
M^{\lambda}_{ab} \to M^{\lambda}_{ab} + g^{\lambda}_{ab}\Xi \qquad \lambda = 2, 3 \tag{3.16}
$$

and a glance to (3.13) shows that the corresponding extensions are always trivial, so the extension coefficients g^{λ}_{ab} are eliminated.

At this point the complete set of Lie brackets of $\overline{sq}_{\omega}(N+1)$ turns out to be

$$
\begin{aligned}
&[J_{ab}, J_{ac}] = \omega_{ab} J_{bc} \qquad [J_{ab}, J_{bc}] = -J_{ac} \qquad [J_{ac}, J_{bc}] = \omega_{bc} J_{ab} \\
&[M^{\alpha}_{ab}, M^{\alpha}_{ac}] = \omega_{ab} J_{bc} \qquad [M^{\alpha}_{ab}, M^{\alpha}_{bc}] = J_{ac} \qquad [M^{\alpha}_{ac}, M^{\alpha}_{bc}] = \omega_{bc} J_{ab} \\
&[J_{ab}, M^{\alpha}_{ac}] = \omega_{ab} M^{\alpha}_{bc} \qquad [J_{ab}, M^{\alpha}_{bc}] = -M^{\alpha}_{ac} \qquad [J_{ac}, M^{\alpha}_{bc}] = -\omega_{bc} M^{\alpha}_{ab} \\
&[M^{\alpha}_{ab}, J_{ac}] = -\omega_{ab} M^{\alpha}_{bc} \qquad [M^{\alpha}_{ab}, J_{bc}] = -M^{\alpha}_{ac} \qquad [M^{\alpha}_{ac}, J_{bc}] = \omega_{bc} M^{\alpha}_{ab} \\
&[J_{ab}, J_{de}] = 0 \qquad [M^{\alpha}_{ab}, M^{\alpha}_{de}] = 0 \qquad [J_{ab}, M^{\alpha}_{de}] = 0 \\
&[J_{ab}, E^{\alpha}_{d}] = (\delta_{ad} - \delta_{bd}) M^{\alpha}_{ab} \qquad [M^{1}_{ab}, E^{1}_{d}] = -(\delta_{ad} - \delta_{bd}) J_{ab} \\
&[M^{\lambda}_{ab}, E^{\lambda}_{d}] = -(\delta_{ad} - \delta_{bd}) J_{ab} \qquad b > a+1
\end{aligned}
\tag{3.17}
$$

$$
[M^{\lambda}_{a\,a+1}, E^{\lambda}_{d}] = -(\delta_{ad} - \delta_{a+1\,d})(J_{a\,a+1} + h^{\lambda}_{a\,a+1}\Xi) \tag{3.18}
$$

$$
[J_{ab}, M^{\alpha}_{ab}] = 2\omega_{ab}(E^{\alpha}_{b} - E^{\alpha}_{a}) + f^{\alpha}_{ab}\Xi \qquad [E^{\alpha}_{a}, E^{\alpha}_{b}] = e^{\alpha}_{a,b}\Xi \qquad a < b \tag{3.19}
$$

$$
\begin{aligned}
&[M^\alpha_{ab}, M^\beta_{ac}] = \omega_{ab}\varepsilon_{\alpha\beta\gamma} M^\gamma_{bc} + \varepsilon_{\alpha\beta\gamma} m^{\alpha,\beta}_{ab,ac}\,\Xi \\
&[M^\alpha_{ab}, M^\beta_{bc}] = \varepsilon_{\alpha\beta\gamma} M^\gamma_{ac} + \varepsilon_{\alpha\beta\gamma} m^{\alpha,\beta}_{ab,bc}\,\Xi \\
&[M^\alpha_{ac}, M^\beta_{bc}] = \omega_{bc}\varepsilon_{\alpha\beta\gamma} M^\gamma_{ab} + \varepsilon_{\alpha\beta\gamma} m^{\alpha,\beta}_{ac,bc}\,\Xi \\
&[M^\alpha_{ab}, M^\beta_{de}] = \varepsilon_{\alpha\beta\gamma} m^{\alpha,\beta}_{ab,de}\,\Xi \\
&[M^\alpha_{ab}, M^\beta_{ab}] = 2\omega_{ab}\varepsilon_{\alpha\beta\gamma}(E^\gamma_a + E^\gamma_b) + \varepsilon_{\alpha\beta\gamma} m^{\alpha,\beta}_{ab}\,\Xi \\
&[M^\alpha_{ab}, E^\beta_d] = (\delta_{ad} + \delta_{bd})\varepsilon_{\alpha\beta\gamma} M^\gamma_{ab} + \varepsilon_{\alpha\beta\gamma} me^{\alpha,\beta}_{ab,d}\,\Xi \\
&[E^\alpha_a, E^\beta_b] = 2\delta_{ab}\varepsilon_{\alpha\beta\gamma} E^\gamma_a + \varepsilon_{\alpha\beta\gamma} e^{\alpha,\beta}_{a,b}\,\Xi.
\end{aligned}
\tag{3.20}
$$

Therefore, the Lie brackets (3.17) are non-extended, the extension coefficients $h^\lambda_{a\,a+1}$ appearing in (3.18) satisfy (3.15), the coefficients of the commutators (3.19) are characterized by the theorem 3.1, and the extension coefficients in the commutators (3.20) are still generic, the redefinitions (3.10) and (3.16) having been already incorporated in the brackets (3.20).

The list of all remaining extension coefficients is

$$
h^\lambda_{a\,a+1} \quad f^\alpha_{ab} \quad e^\alpha_{a,b} \quad
\begin{matrix} m^{\alpha,\beta}_{ab,cd} \\ m^{\alpha,\beta}_{ac,bd} \\ m^{\alpha,\beta}_{ad,bc} \end{matrix} \quad
\begin{matrix} m^{\alpha,\beta}_{ab,ac} \\ m^{\alpha,\beta}_{ab,bc} \\ m^{\alpha,\beta}_{ac,bc} \end{matrix} \quad
m^{\alpha,\beta}_{ab} \quad
\begin{matrix} me^{\alpha,\beta}_{ab,a} \\ me^{\alpha,\beta}_{ab,b} \\ me^{\alpha,\beta}_{ab,e} \end{matrix} \quad
\begin{matrix} e^{\alpha,\beta}_{a,a} \\ e^{\alpha,\beta}_{a,b} \end{matrix}
\tag{3.21}
$$

where the two quaternionic indices α, β are always different, and we assume $a < b < c < d$; e stands for an index different from either a and b.

In the following we proceed to compute the Jacobi identities involving the above coefficients; the results obtained in any equation will be automatically introduced in any further equation, so the order we consider for enforcing the Jacobi identities is an integral part of the derivation, and should be respected. We denote the Jacobi identity (3.2) of the generators X_i, X_j and X_l by $\{X_i, X_j, X_l\}$.

The following equations imply the vanishing of some coefficients:

$$
\begin{aligned}
&\{M^3_{a\,a+1}, E^1_a, E^2_{a+1}\} \qquad h^2_{a\,a+1} = 0 \\
&\{M^2_{a\,a+1}, E^1_a, E^3_{a+1}\} \qquad h^3_{a\,a+1} = 0
\end{aligned}
\tag{3.22}
$$

$$
\{E^\gamma_a, E^\beta_a, E^\alpha_b\} \qquad e^\alpha_{a,b} = 0 \tag{3.23}
$$

$$
\begin{aligned}
&\{M^\alpha_{ab}, M^\alpha_{ac}, E^\gamma_c\} \qquad m^{\alpha,\beta}_{ab,ac} = 0 \\
&\{M^\alpha_{ab}, M^\alpha_{bc}, E^\gamma_c\} \qquad m^{\alpha,\beta}_{ab,bc} = 0 \\
&\{M^\alpha_{ac}, M^\alpha_{bc}, E^\gamma_b\} \qquad m^{\alpha,\beta}_{ac,bc} = 0
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
&\{J_{ab}, M^\beta_{cd}, E^\alpha_b\} \qquad m^{\alpha,\beta}_{ab,cd} = 0 \\
&\{J_{bc}, M^\alpha_{ad}, E^\beta_c\} \qquad m^{\alpha,\beta}_{ad,bc} = 0 \\
&\{J_{ab}, M^\alpha_{bc}, M^\beta_{bd}\} \qquad m^{\alpha,\beta}_{ac,bd} - m^{\beta,\alpha}_{ad,bc} = 0
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&\{M^\beta_{ab}, E^\beta_a, E^\gamma_b\} \qquad me^{\alpha,\beta}_{ab,a} = 0 \\
&\{M^\beta_{ab}, E^\beta_b, E^\gamma_a\} \qquad me^{\alpha,\beta}_{ab,b} = 0
\end{aligned}
\tag{3.26}
$$

$$
\{M^\gamma_{ab}, E^\beta_a, E^\beta_e\} \qquad me^{\alpha,\beta}_{ab,e} = 0 \tag{3.27}
$$

$$
\{E^\alpha_a, E^\alpha_b, E^\gamma_b\} \qquad e^{\alpha,\beta}_{a,b} = 0 \tag{3.28}
$$

so that the only remaining coefficients are f^α_{ab}, $m^{\alpha,\beta}_{ab}$ and $e^{\alpha,\beta}_{a,a}$. The Jacobi identities

$$
\begin{aligned}
&\{J_{ab}, M^\alpha_{ab}, E^\beta_a\} \qquad 2\omega_{ab} e^{\alpha,\beta}_{a,a} - m^{\alpha,\beta}_{ab} + f^\gamma_{ab} = 0 \\
&\{J_{ab}, M^\alpha_{ab}, E^\beta_b\} \qquad 2\omega_{ab} e^{\alpha,\beta}_{b,b} - m^{\alpha,\beta}_{ab} - f^\gamma_{ab} = 0
\end{aligned}
\tag{3.29}
$$

allow us to express the coefficients f_{ab}^{α}, $m_{ab}^{\alpha,\beta}$ in terms of the $e_{a,a}^{\alpha,\beta}$ as follows:

$$\begin{aligned} f_{ab}^{\gamma} &= \omega_{ab}(e_{b,b}^{\alpha,\beta} - e_{a,a}^{\alpha,\beta}) \\ m_{ab}^{\alpha,\beta} &= \omega_{ab}(e_{b,b}^{\alpha,\beta} + e_{a,a}^{\alpha,\beta}). \end{aligned} \tag{3.30}$$

Notice that the first equation is consistent with the relation (3.6). Hence the only Lie brackets of $\overline{sq}_{\omega}(N+1)$ (3.17)–(3.20) which still involve extension coefficients are

$$\begin{aligned} &[J_{ab}, M_{ab}^{\gamma}] = 2\omega_{ab}\{(E_b^{\gamma} + \tfrac{1}{2}e_{b,b}^{\alpha,\beta}\,\Xi) - (E_a^{\gamma} + \tfrac{1}{2}e_{a,a}^{\alpha,\beta}\,\Xi)\} \\ &[M_{ab}^{\alpha}, M_{ab}^{\beta}] = 2\omega_{ab}\varepsilon_{\alpha\beta\gamma}\{(E_a^{\gamma} + \tfrac{1}{2}e_{a,a}^{\alpha,\beta}\,\Xi) + (E_b^{\gamma} + \tfrac{1}{2}e_{b,b}^{\alpha,\beta}\,\Xi)\} \\ &[E_a^{\alpha}, E_a^{\beta}] = 2\varepsilon_{\alpha\beta\gamma}(E_a^{\gamma} + \tfrac{1}{2}e_{a,a}^{\alpha,\beta}\,\Xi). \end{aligned} \tag{3.31}$$

These equations clearly suggest to introduce the redefinition

$$E_a^{\gamma} \to E_a^{\gamma} + \tfrac{1}{2}e_{a,a}^{\alpha,\beta}\,\Xi \tag{3.32}$$

which explicitly shows the triviality of all the extensions determined by the coefficients $e_{a,a}^{\alpha,\beta}$ (and consequently, by all the f_{ab}^{α} and $m_{ab}^{\alpha,\beta}$). Therefore, it is not necessary to compute more Jacobi identities and we can conclude that the most general candidate we started with for a central extension $\overline{sq}_{\omega}(N+1)$ of any algebra in this family is always trivial.

This result can be summed up in the following theorem.

Theorem 3.2. *The second cohomology group $H^2(sq_{\omega}(N+1), \mathbb{R})$ of any Lie algebra belonging to the quaternionic unitary CK family is always trivial, for any N and for any values of the set of constants $\omega_1, \omega_2, \ldots, \omega_N$:*

$$\dim(H^2(sq_{\omega}(N+1), \mathbb{R})) = 0. \tag{3.33}$$

4. Concluding remarks

This paper completes the study of cohomology of the quasi-simple or CK Lie algebras in the three main series (orthogonal, unitary and quaternionic unitary), as associated with anti-Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In contrast to the quasi-orthogonal or quasi-unitary cases, where the dimension of the second cohomology group of a generic algebra in the CK family ranges between zero for the simple algebras and a maximum positive value for the most contracted algebra (with all $\omega_a = 0$), all the central extensions of any of the algebras in the quaternionic quasi-unitary family are always trivial, even for the most contracted algebra. Therefore, from the three types of extensions found in the quasi-orthogonal or quasi-unitary cases, only the first type (extensions which are trivial for all the algebras in the family) is present here. However, we should remark on the suitability of a CK approach to the study of the central extensions of a complete family, because a case-by-case study (for any given algebra in the family) would be no easier than the general analysis we have performed.

In addition to these three *main* families of CK algebras, whose simple members $so(p,q), su(p,q), sq(p,q)$ can be realized as anti-Hermitian matrices over either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, there are other CK families. In the C_{N+1} Cartan series, the remaining real Lie algebra is the real symplectic $sp(2(N+1), \mathbb{R})$, which can be interpreted in terms of CK families either as the single simple member of its own CK family $sp_{\omega_1,\ldots,\omega_N}(2(N+1), \mathbb{R})$, or alternatively and more like the interpretation in this paper, as the unitary family $u_{\omega_1,\ldots,\omega_N}((N+1), \mathbb{H}')$ over the algebra of the split quaternions $\mathbb{H}'$ (a pseudo-orthogonal variant of quaternions, where i_1, i_2, i_3 still anticommute, but their squares are $i_1^2 = -1, i_2^2 = 1, i_3^2 = 1$; this is not a division algebra). The cohomology properties of algebras in this CK family could be studied using an approach similar to that made in this paper for the quaternionic unitary CK algebras. This study, as well as the study of the central extensions of the CK series of the real Lie algebras

$su^*(2r) \approx sl(r, \mathbb{H})$, $so^*(2N)$, $sl(N+1, \mathbb{R}) \approx su(N+1, \mathbb{C}')$ not included in the three main 'signature' series is worth separate consideration.

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